Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 3, pp. 497–501, 1995

0021–8928/95 $24.00+0.00

0021–8928(95)00058-5

THE SOLUTION OF PROBLEMS OF THE THEORY OF ELASTICITY FOR A PLANE WITH A DOUBLY SYMMETRIC TWO-CUSP CUT†

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(*Received* 31 *January* 1994)

The first fundamental and mixed problems of the theory of elasticity for a plane with a doubly symmetric two-cusp cut at the boundary are solved. By changing the parameter this cut can be deformed, bringing the edges together or separating them. The case of free edges and contact displacements in a small neighbourhood of the tip were investigated for a family of cuts of this kind in [1–3]. The first fundamental problem of the theory of elasticity with non-zero boundary stresses is reduced to solving two Hilbert problems for the exterior of the unit circle. A mixed problem when the normal stresses acting at the boundary of the opening ensure specified displacements along a vertical axis. This mixed problem is reduced to a singular Hilbert integral equation and after solving it is reduced to the first fundamental problem of the theory of elasticity solved earlier.

1. SCHEME FOR SOLVING THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY

It is well known [4], that the first fundamental problem of the theory of elasticity for a simply connected region D containing an infinitely distant point reduces to finding two functions that are analytic in D

$$F(z)=\Gamma+\frac{a}{z^2}+\ldots,\quad G(z)=\Gamma'+\frac{a'}{z^2}+\ldots \tag{1.1}$$

where Γ and Γ' are specified ($\operatorname{Im}\Gamma=0$). The boundary condition has the form

$$F(z)+\overline{F(z)}+\exp(-2i\arg z'(t))[z\overline{F'(z)}+\overline{G(z)}]=T_n(t)-iT_s(t) \tag{1.2}$$

where $z=z(t)$, $t\in[t_1,t_2]$ is the equation of the boundary ∂D of the region D, $T_n(t)$ and $T_s(t)$ are the normal and shear components of the external stress, and $\arg z'(t)$ is the angle between the tangent to the contour and the positive direction of the real axis.

We will change to the function $z(\zeta)$, which conformally maps the region $E^-=\{\zeta=\xi+i\eta\,|\,|\zeta|>1\}$ into D with corresponding $z(\infty)=\infty$. The problem then reduces to a boundary-value problem in the region E^- with the following boundary condition

$$\{2\operatorname{Re}\Phi(\zeta)-\exp(2i\arg[\zeta z'(\zeta)])\chi(\zeta)\}\big|_{\zeta=e^{i\theta}}=T_n(t(\theta))+iT_s(t(\theta)) \tag{1.3}$$

Here

$$\chi(\zeta)=\overline{z(\zeta)}\frac{\Phi'(\zeta)}{z'(\zeta)}+\Psi(\zeta),\quad \Phi(\zeta)=F(z(\zeta))$$

$$\Psi(\zeta)=G(z(\zeta)),\quad z(t(\theta))=z(e^{i\theta}),\quad \theta\in[-\pi,\pi]$$

$$\Phi(\zeta)=\Gamma+\frac{b}{\zeta^2}+\ldots,\quad \Psi(\zeta)=\Gamma'+\frac{b'}{\zeta^2}+\ldots,\quad \zeta\in E^- \tag{1.4}$$

The complex relation (1.3) can be written as two real relations

$$\operatorname{Re}\Phi(e^{i\theta})=\frac{1}{2}|z'(e^{i\theta})|^{-2}\operatorname{Re}\{\zeta^2z'^2(\zeta)\chi(\zeta)\}\Big|_{\zeta=e^{i\theta}}+\frac{1}{2}T_n(t(\theta)) \tag{1.5}$$

†*Prikl. Mat. Mekh.* Vol. 59, No. 3, pp. 524–528, 1995.

$$\mathrm{Im}\{\zeta^2 z'^2(\zeta)\chi(\zeta)\}\Big|_{\zeta=e^{i\theta}} = -|z'(e^{i\theta})|^2 T_s(t(\theta)) \tag{1.6}$$

If, after substituting ζ^{-1} instead of ζ in the expression for $\overline{z(\zeta)}$ we obtain a function $q(\zeta)$ that is meromorphic in E^-, the function

$$K(\zeta) = \zeta^2 z'^2(\zeta)\left[q(\zeta)\frac{\Phi'(\zeta)}{z'(\zeta)} + \Psi(\zeta)\right], \quad \zeta \in E^- \tag{1.7}$$

will also be meromorphic in E^-, where

$$K(e^{i\theta}) = \{\zeta^2 z'^2(\zeta)\chi(\zeta)\}\Big|_{\zeta=e^{i\theta}} \tag{1.8}$$

According to relation (1.6) the function $K(\zeta)$ can be re-established from the boundary value of its imaginary part. The expression for $K(\zeta)$ will contain unknown constants, the number of which is determined by the number and order of the poles of the function $K(\zeta)$ [5, p. 270]. Further, $\Phi(\zeta)$ can be recovered in accordance with (1.5) from the boundary value of its real part. To determine the unknown constants we will use the equation $\Phi(\infty) = \Gamma$, the condition $\mathrm{Res}_\infty \Phi(\zeta) = 0$, and the condition for $\Psi(\zeta)$ to be analytic in E^-. Here $T_n(t(\theta))$ and $T_s(t(\theta))$ are assumed to satisfy the Hölder condition.

2. SOLUTION OF THE PROBLEM

Consider the function

$$z(\zeta) = \frac{i(b^2\zeta^2+1)}{\zeta(b^2-1)} + \frac{\zeta(b^2-1)}{i(b^2\zeta^2+1)}, \quad b>1, \ \zeta \in E^- \tag{2.1}$$

which, for varying b, maps the region E^- into the regions D_b, which are the exteriors of regions, symmetrical about the axes of coordinates, with cusps at the points $z_{1,2} = \pm 2$. As $b \to \infty$ we obtain a plane with a cut along the section $[-2, 2]$. The edges of the cut diverge when $1 < b < \infty$ (the greatest distance between them is $8b^2(b^4-1)^{-1}$), closing up at the points ± 2 at zero angles.

The function $K(\zeta)$ from (1.7), by relations (1.6), (1.8) and the expansion (1.4), has the form

$$K(\zeta) = A - \frac{\Gamma' b^4}{(b^2-1)^2}\zeta^2 - \frac{\overline{\Gamma'} b^4}{(b^2-1)^2}\zeta^{-2} + C\zeta + \overline{C}\zeta^{-1} + D\frac{1+ib\zeta}{\zeta-ib} + \overline{D}\frac{\zeta-ib}{1+ib\zeta} + E\frac{1-ib\zeta}{\zeta+ib} +$$

$$+\overline{E}\frac{\zeta+ib}{1-ib\zeta} + \frac{1}{2\pi i}\int_{-\pi}^{\pi} R(\tau)\cos^2\tau \,\mathrm{ctg}\frac{\tau-\theta}{2}\,d\tau \tag{2.2}$$

$$\mathrm{Im}\,A = 0, \quad R(\tau) = \frac{|1-b^2e^{2i\tau}|^2\,|b^4e^{2i\tau}+1|^2}{|b^2e^{2i\tau}+1|^2} T_s(t(\tau))$$

The function $\Phi(\zeta)$, recovered using condition (1.5), will contain the same unknown constants. However, two of them (A and $\mathrm{Im}\,C$) can be eliminated by utilizing, as in [4, p. 440], the requirement that the function $F(z)$ has singularities of order less than unity at the cut tips.

Hence

$$\Phi(\zeta) = \frac{\alpha+i\beta}{\zeta-i} + \frac{\gamma+i\delta}{\zeta+i} + \Phi_0(\zeta) + \frac{1}{2\pi}\int_{-\pi}^{\pi} Q(\theta)\frac{\zeta+e^{i\theta}}{\zeta-e^{i\theta}}\,d\theta \tag{2.3}$$

where

$$\alpha = \left[2b^4\,\mathrm{Im}\,\Gamma' + (b^2-1)^2\,\mathrm{Re}\,C + (1-b^2)[(b+1)^2\,\mathrm{Re}\,D + (b-1)^2\,\mathrm{Re}\,E] + \right.$$

$$\left. + \frac{1}{\pi}\int_{-\pi}^{\pi} R(\tau)\cos\tau\,\mathrm{ctg}\frac{\tau-\frac{\pi}{2}}{2}\,d\tau\right](1+b^2)^{-4}2^{-2}$$

$$\gamma = \left[-2b^4\,\mathrm{Im}\,\Gamma' + (b^2-1)^2\,\mathrm{Re}\,C + (1-b^2)[(b-1)^2\,\mathrm{Re}\,D + (1+b)^2\,\mathrm{Re}\,E] + \right.$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi} R(\tau)\cos\tau \operatorname{ctg}\frac{\tau+\frac{\pi}{2}}{2}d\tau\Bigg](1+b^2)^{-4}2^{-2} \tag{2.4}$$

$$\Phi_0(\zeta)=\frac{\beta}{2}-\frac{\delta}{2}-\frac{ib^4\operatorname{Im}\Gamma'}{(1+b^2)^3(1+b^6)}\left[\frac{1+b^4}{1-\zeta^2b^2}-\frac{1+b^2+b^4}{1+\zeta^2b^4}\right]+\frac{\zeta b^2(b^2-1)^2\operatorname{Re}C}{(1+b^2)^2(1+b^6)(1-b^8)}\times$$

$$\times\left[\frac{2(1+b^4)^2}{1-\zeta^2b^2}+\frac{(1-b^6)b^2}{1+\zeta^2b^4}\right]-\frac{b^4\operatorname{Re}\Gamma'}{2(1+b^2)(b^6+1)(1-b^8)}\times$$

$$\times\left[-1-b^2-4b^4-b^6-b^8+\frac{4(1+b^4)^2}{1-\zeta^2b^2}-\frac{2(b^2-1)(b^6-1)}{1+\zeta^2b^4}\right]+\frac{(1-b^2)b^2}{(1+b^2)^2}\times$$

$$\times\left\{\left[\frac{b^6(1-b^2)(b^4-1)}{(1+b^2\zeta^2)(1+b^6)}+\frac{(1-b^4)(1+b^2)}{(1+b^6)(1-b^2\zeta^2)}-\frac{b^4(1+b^2)^2}{(1+b^4)(1+b^4\zeta^2)}\right]\zeta(\operatorname{Re}D+\operatorname{Re}E)+\right.$$

$$\left.+\left[\frac{b^5(1-b^2)(b^4-1)}{(1+b^2\zeta^2)(1+b^6)}-\frac{(1-b^2)(1-b^4)}{(1+b^6)(1-b^2\zeta^2)b}-\frac{(1+b^2)b}{1+ib^4\zeta^2}\right]i(\operatorname{Re}D-\operatorname{Re}E)\right\}+\frac{b(b^2-1)^2}{2}\times$$

$$\times\left\{\left[\frac{1+b^2+b^4}{(b^8-1)(b^6+1)}+\frac{2}{(1+b^6)(1-b^4)(1-\zeta^2b^2)}-\frac{2b^2}{(1+b^6)(b^8-1)(1+\zeta^2b^4)}\right](\operatorname{Im}E-\operatorname{Im}D)+\right.$$

$$\left.+\left[\frac{1}{(1+b^2)^2(1-\zeta^2b^2)}-\frac{2}{(1+b^2)(b^4-1)(1+\zeta^2b^4)}\right]\frac{(\operatorname{Im}D+\operatorname{Im}E)bi\zeta}{1+b^6}\right\} \tag{2.5}$$

$$Q(\theta)=\frac{1}{4\pi}\int_{-\pi}^{\pi}R(\tau)\left\{|b^2e^{2i\theta}+1|^2(1+b^2)^4 ctg\frac{\tau-\theta}{2}-4(\sin\tau+\sin\theta)b^2\cos\theta\times\right.$$

$$\left.\times[(1+b^2)^4+(1+b^8+2b^4\cos2\theta)-b^2(1+b^2)^2]\right\}d\tau|1-b^2e^{2i\theta}|^{-2}|b^2+1|^{-4}+\frac{1}{2}T_n(t(\theta)) \tag{2.6}$$

To determine the constants α, β, γ, δ, Re C, Re D, Im D, Re E and Im E, in addition to relations (2.4) we use the conditions $\Phi(\infty)=\Gamma$, $\operatorname{Res}_\infty\Phi(\zeta)=0$, $\operatorname{Res}_{\pm ib}\Psi(\zeta)=0$, which give the following equations

$$(\operatorname{Im}E-\operatorname{Im}D)\frac{b(b^6-1)}{2(b^2+1)(b^4+1)(b^6+1)}=\Gamma+\frac{\operatorname{Re}\Gamma' b^4(1+b^2+4b^4+b^6+b^8)}{2(b^6+1)(b^2+1)(b^8-1)}-\frac{1}{2\pi}\int_{-\pi}^{\pi}Q(\theta)d\theta$$

$$\alpha+i\beta+\gamma+i\delta+\frac{\operatorname{Re}C(b^2-1)^2[(1-b^6)-2(1+b^4)^2]}{(1+b^2)^2(1+b^6)(1-b^8)}+\frac{b^2(1-b^2)}{(1+b^2)^2}\times$$

$$\times\left[\frac{b^4(1-b^2)(b^4-1)}{1+b^6}+\frac{(b^4-1)(1+b^2)}{b^2(1+b^6)}-\frac{(1+b^2)^2}{1+b^4}\right](\operatorname{Re}D+\operatorname{Re}E)-$$

$$-\frac{ib^2(b^2-1)^2(\operatorname{Im}D+\operatorname{Im}E)}{(1+b^2)(1+b^6)(b^4-1)}=-\frac{1}{\pi}\int_{-\pi}^{\pi}Q(\theta)e^{i\theta}d\theta$$

$$D+\frac{(b^6-1)(b^4+1)}{2(b^4-1)^2}\left[\frac{\alpha+i\beta}{(b-1)^2}+\frac{\gamma+i\delta}{(b+1)^2}+\Phi_0'(ib)\right]=\frac{(b^6-1)(b^4+1)}{(b^4-1)^2 2\pi}\int_{-\pi}^{\pi}Q(\theta)\frac{e^{i\theta}d\theta}{(ib-e^{i\theta})^2} \tag{2.7}$$

$$E+\frac{(b^6-1)(b^4+1)}{2(b^4-1)^2}\left[\frac{\alpha+i\beta}{(b+1)^2}+\frac{\gamma+i\delta}{(b-1)^2}+\Phi_0'(-ib)\right]=\frac{(b^6-1)(b^4+1)}{(b^4-1)^2 2\pi}\int_{-\pi}^{\pi}Q(\theta)\frac{e^{i\theta}d\theta}{(ib-e^{i\theta})^2}$$

where $\Phi_0(\zeta)$ is given by (2.5). Hence, we have nine real linear equations connecting nine unknown constants. After determining these constants, $\Phi(\zeta)$ is established from (2.3), (2.4) and (2.6), and $\Psi(\zeta)$ is found from (1.7) and (2.2).

3. THE MIXED PROBLEM

We will denote by $f(z)$ and $g(z)$ functions, analytic in D_b, with a pole at infinity such that $f'(z)=F(z)$, $g'(z)=G(z)$.

Suppose $T_s(t) \equiv 0, z(t) \in \partial D_b$. In addition, the vertical component of the boundary displacement is specified, namely

$$\operatorname{Im}[kf(z) - z\overline{F(z)} - \overline{g(z)}]\Big|_{z=z(t)} = 2\mu\upsilon(t), \quad k > 1 \tag{3.1}$$

As above, we change to the region E^-. The function $\Phi(\zeta) = F(z(\zeta))$ has the form (2.3), where $T_n(t(\theta))$ is an unknown function and $T_s(t(\theta)) \equiv 0$. Assuming that $\upsilon(t)(\theta))$ has a Hölder derivative with respect to θ and differentiating both sides of (3.1) with respect to θ, we obtain

$$\operatorname{Im}\{ie^{i\theta}z'(e^{i\theta})[(k+1)\Phi(e^{i\theta}) - T_n(t(\theta))]\} = 2\mu\frac{d\upsilon(t(\theta))}{d\theta}$$

Consequently, to determine $T_n(t(\theta))$ we have the Hilbert equation [5, p. 292]

$$\operatorname{Re}\left\{e^{i\theta}z'(e^{i\theta})\left[(k-1)T_n(t(\theta)) + \frac{i(k+1)}{2\pi}\int_{-\pi}^{\pi} T_n(t(\sigma))\operatorname{ctg}\frac{\sigma-\theta}{2}d\sigma\right]\right\} = 2r(\theta) \tag{3.2}$$

where

$$r(\theta) = 2\mu\frac{d\upsilon(t(\theta))}{d\theta} - (k+1)\operatorname{Re}\left\{e^{i\theta}z'(e^{i\theta})\left[\frac{\alpha+i\beta}{\zeta-i} + \frac{\gamma+i\delta}{\zeta+i} + \Phi_0(e^{i\theta})\right]\right\}$$

$z(\zeta)$ is found from (2.1), and α, β, γ, δ and $\Phi_0(\zeta)$ are found from (2.3) and (2.5).

We obtain the regularizing factor—a real-valued function, after multiplying both sides of Eq. (3.2) by which we obtain the problem of establishing an analytic function from the boundary values of its real part [5, pp. 275, 292]

$$p(\theta) = \exp\left\{-\frac{1}{2\pi}\int_{-\pi}^{\pi} A(\tau)\operatorname{ctg}\frac{\tau-\theta}{2}d\tau\right\}[(k-1)^2\operatorname{Re}^2\varphi(e^{i\theta}) + (k+1)^2\operatorname{Im}^2\varphi(e^{i\theta})]^{1/2}$$

$$A(\tau) = \operatorname{arctg}\left[\frac{(k+1)\operatorname{Im}\varphi(e^{i\tau})}{(k-1)\operatorname{Re}\varphi(e^{i\tau})}\right]$$

$$\operatorname{Re}\varphi(e^{i\theta}) = -(b^2+1)b^2[b^2\sin 4\theta + (3b^4 - 4b^2 + 3)\sin 2\theta]/(b^2-1)\Delta$$

$$\operatorname{Im}\varphi(e^{i\theta}) = [b^4\cos 4\theta + b^2(1+b^2)^2\cos 2\theta + 1 + b^2 - b^4 + b^6 + b^8]/\Delta$$

$$\Delta = (1 + b^4 + 2b^2\cos 2\theta)^2$$

Re-establishing the corresponding function, we obtain

$$T_n(t(\theta)) = \operatorname{Re}\frac{2N\sin\theta + iL + s(\theta) + \dfrac{i}{2\pi}\displaystyle\int_{-\pi}^{\pi} s(\sigma)\operatorname{ctg}\frac{\sigma-\theta}{2}d\sigma}{2q(e^{i\theta})\cos\theta} \tag{3.3}$$

$$q(\zeta) = \exp\left\{-\frac{1}{2\pi i}\int_{-\pi}^{\pi} A(\theta)\frac{\zeta+e^{i\theta}}{\zeta-e^{i\theta}}d\theta\right\}, \quad s(\theta) = r(\theta)\cdot p(\theta)$$

Here N and L are arbitrary real constants.

The right-hand side of (3.3) depends linearly both on the constants N and L and, because of the function $s(\theta)$, on the constants α, β, γ, δ, Re C, Re D, Im D, Re E and Im E. Substituting $T_n(t(\theta))$ from (3.3) into the right-hand side of system (2.4), (2.7) instead of $2Q(\theta)$ and adding the complex equation

$$\int_{-\pi}^{\pi} T_n(t(\tau))z'(e^{i\tau})e^{i\tau}d\tau = 0$$

with $T_n(t(\tau))$ from (3.3), we obtain a linear system of 11 real equations with 11 unknowns. Solving it we obtain $\Phi(\zeta)$ and $\Psi(\zeta)$, as in the previous problem.

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Translated by R.C.G.